Schwartz 8.2

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2$$

$$\gamma_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \partial_\nu A_\alpha - g_{\mu\nu} \mathcal{L}$$

$$= -F_{\mu\alpha} \partial_\nu A_\alpha + g_{\mu\nu} \frac{1}{4} F_{\alpha\beta}^2$$

$$\mathcal{E} = \gamma_{00} = -F_{0\alpha} \partial_0 A_\alpha + g_{00} \left(\frac{1}{4} F_{\alpha\beta}^2\right)$$

This Lagrangian has gauge symmetry

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

use this gauge symmetry to set $A_0 = 0$.

$$\Rightarrow \gamma_{00} = \mathcal{E} = +F_{0i}\, \partial_0 A_i + \frac{1}{4} F_{\alpha\beta}^2$$

$$= (\partial_0 \vec{A} - \vec{\nabla} A_0) \cdot \partial_0 \vec{A} + \frac{1}{2}(\vec{B}^2 - \vec{E}^2)$$

Recall $\vec{E} = \partial_0 \vec{A} - \vec{\nabla} A_0$, $\vec{B} = \vec{\nabla} \times \vec{A}$, $-\frac{1}{4} F_{\alpha\beta}^2 = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$

$$\Rightarrow \quad \mathcal{E} = (\partial_0 \vec{A} - \vec{\nabla} A_0) \cdot (\partial_0 \vec{A} - \vec{\nabla} A_0) + \frac{1}{2}(\vec{B}^2 - \vec{E}^2)$$
$$+ (\partial_0 \vec{A} - \vec{\nabla} A_0) \cdot \vec{\nabla} A_0$$

$$= \vec{E}^2 + \frac{1}{2}(\vec{B}^2 - \vec{E}^2) = \boxed{\frac{1}{2}(\vec{E}^2 + \vec{B}^2)}$$

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2.21.2024